

Appendix 2

For a two port network as shown in Figure 8 the Z-parameters can be calculated as follows,

$$\begin{aligned}
 Z_{11} &= r_1 + \frac{\frac{1}{g_1 + j\omega c_1} \left( Z + \frac{1}{g_2 + j\omega c_2} \right)}{\frac{1}{g_1 + j\omega c_1} + Z + \frac{1}{g_2 + j\omega c_2}} \\
 &= r_1 + \frac{(g_2 + j\omega c_2)Z + 1}{g_1 + g_2 + j\omega c_1 + j\omega c_2 + Z(g_1 + j\omega c_1)(g_2 + j\omega c_2)} \\
 &= r_1 + \frac{(g_2 + j\omega c_2)Z + 1}{\Delta}
 \end{aligned} \tag{7}$$

$$\begin{aligned}
 Z_{12} &= \frac{\frac{1}{g_1 + j\omega c_1} \frac{1}{g_2 + j\omega c_2}}{\frac{1}{g_1 + j\omega c_1} + Z + \frac{1}{g_2 + j\omega c_2}} \\
 &= \frac{1}{(g_1 + j\omega c_1)(g_2 + j\omega c_2) \left( \frac{1}{g_1 + j\omega c_1} + Z + \frac{1}{g_2 + j\omega c_2} \right)} \\
 &= \frac{1}{g_1 + g_2 + j\omega c_1 + j\omega c_2 + Z(g_1 + j\omega c_1)(g_2 + j\omega c_2)} \\
 &= \frac{1}{\Delta}
 \end{aligned} \tag{8}$$

$$\begin{aligned}
 Z_{22} &= r_2 + \frac{\frac{1}{g_2 + j\omega c_2} \left( Z + \frac{1}{g_1 + j\omega c_1} \right)}{\frac{1}{g_1 + j\omega c_1} + Z + \frac{1}{g_2 + j\omega c_2}} \\
 &= r_2 + \frac{(g_1 + j\omega c_1)Z + 1}{g_1 + g_2 + j\omega c_1 + j\omega c_2 + Z(g_1 + j\omega c_1)(g_2 + j\omega c_2)} \\
 &= r_2 + \frac{(g_1 + j\omega c_1)Z + 1}{\Delta}
 \end{aligned} \tag{9}$$

From (8) we can get,

$$Z(g_1 + j\omega c_1)(g_2 + j\omega c_2) = \frac{1}{Z_{12}} - g_1 - g_2 - j\omega c_1 - j\omega c_2 \tag{10}$$

Rewritten (7) as:

$$Z_{11} = r_1 + [(g_2 + j\omega c_2)Z + 1]Z_{12}$$

multiply  $(g_1 + j\omega c_1)$ , if  $(g_1 + j\omega c_1) \neq 0$ , we get

$$(g_1 + j\omega c_1)Z_{11} = r_1(g_1 + j\omega c_1) + (g_1 + j\omega c_1)[(g_2 + j\omega c_2)Z + 1]Z_{12} \tag{11}$$

$$= r_1(g_1 + j\omega c_1) + [(g_1 + j\omega c_1)(g_2 + j\omega c_2)Z]Z_{12} + (g_1 + j\omega c_1)Z_{12}$$

substitute (10) to (11) to eliminate Z,

$$\begin{aligned}
 (g_1 + j\omega c_1)Z_{11} &= r_1(g_1 + j\omega c_1) + \left(\frac{1}{Z_{12}} - g_1 - g_2 - j\omega c_1 - j\omega c_2\right)Z_{12} + (g_1 + j\omega c_1)Z_{12} \\
 &= r_1(g_1 + j\omega c_1) + 1 - (g_1 + j\omega c_1)Z_{12} - (g_2 + j\omega c_2)Z_{12} + (g_1 + j\omega c_1)Z_{12} \quad (12) \\
 &= r_1(g_1 + j\omega c_1) + 1 - (g_2 + j\omega c_2)Z_{12}
 \end{aligned}$$

Similarly we can get an equation about  $Z_{22}$  as

$$(g_2 + j\omega c_2)Z_{22} = r_2(g_2 + j\omega c_2) + 1 - (g_1 + j\omega c_1)Z_{12} \quad (13)$$

We write the Z-parameters in their real and imaginary part as

$$\begin{aligned}
 Z_{11} &= Z_{11a} + jZ_{11b} \\
 Z_{12} &= Z_{12a} + jZ_{12b} \\
 Z_{22} &= Z_{22a} + jZ_{22b}
 \end{aligned} \quad (14)$$

Substitute (14) to (12) we get

$$\begin{aligned}
 (g_1 + j\omega c_1)(Z_{11a} + jZ_{11b}) &= r_1(g_1 + j\omega c_1) + 1 - (g_2 + j\omega c_2)(Z_{12a} + jZ_{12b}) \\
 g_1 Z_{11a} - \omega c_1 Z_{11b} + j(g_1 Z_{11b} + Z_{11a} \omega c_1) &= r_1 g_1 + 1 - g_2 Z_{12a} + \omega c_2 Z_{12b} + j(r_1 \omega c_1 - \omega c_2 Z_{12a} - g_2 Z_{12b}) \quad (15)
 \end{aligned}$$

By separate real and imaginary part, we get two equations,

$$g_1 Z_{11a} - \omega c_1 Z_{11b} - r_1 g_1 - 1 + g_2 Z_{12a} - \omega c_2 Z_{12b} = 0 \quad (16)$$

$$g_1 Z_{11b} + Z_{11a} \omega c_1 - r_1 \omega c_1 + \omega c_2 Z_{12a} + g_2 Z_{12b} = 0 \quad (17)$$

Be substituting (14) to (13) we can get another two equations in a similar way,

$$g_2 Z_{22a} - \omega c_2 Z_{22b} - r_2 g_2 - 1 + g_1 Z_{12a} - \omega c_1 Z_{12b} = 0 \quad (18)$$

$$g_2 Z_{22b} + Z_{22a} \omega c_2 - r_2 \omega c_2 + \omega c_1 Z_{12a} + g_1 Z_{12b} = 0 \quad (19)$$

When the measurements are taken at two frequencies, we can get another set of equation at frequency  $\omega_2$ , if we denote the measurements at this frequency by adding a superscript 2 to the corresponding quantities, the equations can be written as follows

$$g_1 Z_{11a}^2 - \omega_2 c_1 Z_{11b}^2 - r_1 g_1 - 1 + g_2 Z_{12a}^2 - \omega_2 c_2 Z_{12b}^2 = 0 \quad (20)$$

$$g_1 Z_{11b}^2 + Z_{11a}^2 \omega_2 c_1 - r_1 \omega_2 c_1 + \omega_2 c_2 Z_{12a}^2 + g_2 Z_{12b}^2 = 0 \quad (21)$$

$$g_2 Z_{22a}^2 - \omega_2 c_2 Z_{22b}^2 - r_2 g_2 - 1 + g_1 Z_{12a}^2 - \omega_2 c_1 Z_{12b}^2 = 0 \quad (22)$$

$$g_2 Z_{22b}^2 + Z_{22a}^2 \omega_2 c_2 - r_2 \omega_2 c_2 + \omega_2 c_1 Z_{12a}^2 + g_1 Z_{12b}^2 = 0 \quad (23)$$

The problem is to solve equations (16) to (23) for model parameters  $r_k$ ,  $g_k$  and  $c_k$ ,  $k=1,2$ .

To eliminate  $r_1$  and  $r_2$ , first let (16) - (20), we get,

$$g_1(Z_{11a} - Z_{11a}^2) - c_1(\omega Z_{11b} - \omega_2 Z_{11b}^2) + g_2(Z_{12a} - Z_{12a}^2) - c_2(\omega Z_{12b} - \omega_2 Z_{12b}^2) = 0 \quad (24)$$

then  $\omega_2 \times (17) - \omega \times (21)$  which gives,

$$g_1(\omega_2 Z_{11b} - \omega Z_{11b}^2) - \omega \omega_2 c_1(Z_{11a} - Z_{11a}^2) + g_2(\omega_2 Z_{12b} - \omega Z_{12b}^2) - \omega \omega_2 c_2(Z_{12a} - Z_{12a}^2) = 0 \quad (25)$$

similarly, (12) - (16) yields,

$$g_2(Z_{22a} - Z_{22a}^2) - c_2(\omega Z_{22b} - \omega_2 Z_{22b}^2) + g_1(Z_{12a} - Z_{12a}^2) - c_1(\omega Z_{12b} - \omega_2 Z_{12b}^2) = 0 \quad (26)$$

Let,

$$a_1 = Z_{11a} - Z_{11a}^2 \quad (27)$$

$$a_2 = Z_{12a} - Z_{12a}^2 \quad (28)$$

$$a_3 = Z_{22a} - Z_{22a}^2 \quad (29)$$

and

$$b_1 = \omega Z_{11b} - \omega_2 Z_{11b}^2 \quad (30)$$

$$b_2 = \omega_2 Z_{11b} - \omega Z_{11b}^2 \quad (31)$$

$$b_3 = \omega Z_{12b} - \omega_2 Z_{12b}^2 \quad (32)$$

$$b_4 = \omega_2 Z_{12b} - \omega Z_{12b}^2 \quad (33)$$

$$b_5 = \omega Z_{22b} - \omega_2 Z_{22b}^2 \quad (34)$$

$$b_6 = \omega_2 Z_{22b} - \omega Z_{22b}^2 \quad (35)$$

So (24) to (26) can be rewritten as

$$a_1 g_1 - b_1 c_1 + a_2 g_2 - b_3 c_2 = 0 \quad (36)$$

$$b_2 g_1 + a_1 \omega \omega_2 c_1 + b_4 g_2 + a_2 \omega \omega_2 c_2 = 0 \quad (37)$$

$$a_3 g_2 - b_5 c_2 + a_2 g_1 - b_3 c_1 = 0 \quad (38)$$

Solve (36) to (38) for  $g_1$ ,  $c_1$ , and  $c_2$  we get,

$$g_1 = - \frac{a_1 a_2 b_3 \omega \omega_2 + b_1 b_4 b_5 - (b_3)^2 b_4 + a_2 a_3 b_1 \omega \omega_2 - a_1 a_3 b_3 \omega \omega_2 - (a_2)^2 b_3 \omega \omega_2}{\Delta_1} g_2 \quad (39)$$

$$c_1 = - \frac{a_1 b_4 b_5 + a_1 a_2 a_3 \omega \omega_2 - a_2 b_2 b_5 - (a_2)^3 \omega \omega_2 - a_2 b_2 b_4 + a_2 b_2 b_3}{\Delta_1} g_2 \quad (40)$$

$$c_2 = \frac{-a_2 b_1 b_4 + a_3 b_1 b_2 - a_1 (a_2)^2 \omega \omega_2 - a_2 b_2 b_3 + (a_1)^2 a_3 \omega \omega_2 + a_1 b_3 b_4}{\Delta_1} g_2 \quad (41)$$

where

$$\Delta_1 = b_1 b_2 b_3 + (a_1)^2 b_3 \omega \omega_2 - b_2 (b_3)^2 + (a_2)^2 b_1 \omega \omega_2 - 2 a_1 a_2 b_3 \omega \omega_2 \quad (42)$$

If we denote the coefficient of (39)-(41) by  $n_j$ ,  $j=1, 2, 3$ , we get,

$$g_1 = n_1 g_2 \quad (43)$$

$$c_1 = n_2 g_2 \quad (44)$$

$$c_2 = n_3 g_2 \quad (45)$$

where

$$n_1 = - \frac{a_1 a_2 b_3 \omega \omega_2 + b_1 b_4 b_5 - (b_3)^2 b_4 + a_2 a_3 b_1 \omega \omega_2 - a_1 a_3 b_3 \omega \omega_2 - (a_2)^2 b_3 \omega \omega_2}{\Delta_1} \quad (46)$$

$$n_2 = - \frac{a_1 b_4 b_5 + a_1 a_2 a_3 \omega \omega_2 - a_2 b_2 b_5 - (a_2)^3 \omega \omega_2 - a_2 b_2 b_4 + a_2 b_2 b_3}{\Delta_1} \quad (47)$$

$$n_3 = \frac{-a_2 b_1 b_4 + a_3 b_1 b_2 - a_1 (a_2)^2 \omega \omega_2 - a_2 b_2 b_3 + (a_1)^2 a_3 \omega \omega_2 + a_1 b_3 b_4}{\Delta_1} \quad (48)$$

Substitute  $g_1, c_1, c_2$  to (17)

$$n_1 g_2 Z_{11b} + Z_{11a} \omega n_2 g_2 - r_1 \omega n_2 g_2 + \omega n_3 g_2 Z_{12a} + g_2 Z_{12b} = 0 \quad (49)$$

if  $g_2 \neq 0$ , we can solve (49) for  $r_1$  as follows

$$r_1 = \frac{n_1 Z_{11b} + Z_{11a} \omega n_2 + \omega n_3 Z_{12a} + Z_{12b}}{\omega n_2} \quad (50)$$

Similarly we can get  $r_2$  from (19)

$$r_2 = \frac{n_1 Z_{11b} + Z_{11a} \omega n_2 + \omega n_3 Z_{12a} + Z_{12b}}{\omega n_2} \quad (51)$$

Substitute  $g_1, c_1, c_2$  and  $r_1$  to (16), we can determine  $g_2$ .

$$g_2 = - \frac{\omega n_2}{(n_1)^2 Z_{11b} + n_1 n_3 Z_{12a} \omega + n_1 Z_{12b} - n_2 Z_{12a} \omega + n_2 n_3 Z_{12b} (\omega)^2 + (n_2)^2 Z_{11b} (\omega)^3} \quad (52)$$

After  $g_2$  is determined,  $g_1, c_1, c_2$  can be calculated from (43) to (45). So far, the only parameter left undetermined is  $Z$ , which can be calculated at frequency  $\omega$  from (10) as

$$Z = \frac{\frac{1}{Z_{12}} - g_1 - g_2 - j\omega c_1 - j\omega c_2}{(g_1 + j\omega c_1)(g_2 + j\omega c_2)} \quad (53)$$